Homotopy Analysis Method for Nonlinear Differential Equations with Fractional Orders

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The aim of this paper is to solve nonlinear differential equations with fractional derivatives by the homotopy analysis method. The fractional derivative is described in Caputo's sense. It shows that the homotopy analysis method not only is efficient for classical differential equations, but also is a powerful tool for dealing with nonlinear differential equations with fractional derivatives.

Key words: Nonlinear Differential Equation; Homotopy Analysis; Fractional Derivative.

1. Introduction

Most problems in science and engineering are nonlinear. Thus, it is important to develop efficient methods to solve them. In the past decades, with the fast development of high-quality symbolic computing software, such as Maple, Mathematica and Matlab, analytic as well as numerical techniques for nonlinear differential equations have been developed quickly. The homotopy analysis method (HAM) [1 – 5] is one of the most effective methods to construct analytically approximate solutions of nonlinear differential equations. This method has been applied to a wide range of nonlinear differential equations. Compared with the traditional analytic approximation tools, such as the perturbation method [6 – 9], the δ-expansion method [10], and the Adomian decomposition method [11 – 13], the HAM provides a convenient way to control and adjust the convergence range and the rate of approximation. Also, the HAM is valid even if a nonlinear problem does not contain a small or large parameter. In addition, it can be employed to approximate a nonlinear problem by choosing different sets of base functions.

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in physics and engineering [14]. For instance for the propagation of waves through a fractal medium or diffusion in a disordered system it is reasonable to formulate the structure of the nonlinear evolution equations in terms of fractional derivatives rather than in the classical form. Furthermore, we known that many nonlinear differential equations exhibit strange attractors and their solutions have been discovered to move toward strange attractors [15]. Such strange attractors are fractals by definition. We therefore aim to deal with fractal nonlinear differential equations rather than with classical forms of them. In this paper, we employ the HAM to solve fractional nonlinear differential equations. Some examples are used to illustrate the effectiveness of this method. It is shown that the HAM is efficient not only for classical differential equations but also for differential equations with fractional derivatives.

2. Fractional Integration and Differentiation

In this section, let us first recall essentials of the fractional calculus. Fractional calculus is the name of the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. There are many books [14, 16 – 18] that develop the fractional calculus and various definitions of fractional integration and differentiation, such as Grunwald-Letnikov's definition, Riemann-Liouville's definition, Caputo's definition and the generalized function approach. For the purpose of this paper, the Caputo derivative as well as the Riemann-Liouville integral will be used.

Let $\mathscr{D}_*{}^{\alpha}$ denote the differential operator in the sense of Caputo [19], defined by

$$\mathscr{D}_*{}^{\alpha} f(x) = \mathscr{J}^{m-\alpha} \mathscr{D}^m f(x), \tag{2.1}$$

where $m-1 < \alpha \leq m$, f is a (in general nonlinear) function, $\mathscr{D}^m$ the usual integer differential operator of order m and $\mathscr{J}^{\alpha}$ the Riemann-Liouville integral operator of order $\alpha > 0$, defined by

$$\mathscr{J}^{\alpha} f(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\alpha-1} f(t) \mathrm{d}t, \quad x > 0. \quad (2.2)$$

Properties of the operator $\mathscr{J}^{\alpha}$ can be found in [16, 18]. We mention only the following:

For $f \in C_{\mu}, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$:

(1) $\mathscr{J}^{\alpha} \mathscr{J}^{\beta} f(x) = \mathscr{J}^{\alpha+\beta} f(x)$;
(2) $\mathscr{J}^{\alpha} \mathscr{J}^{\beta} f(x) = \mathscr{J}^{\beta} \mathscr{J}^{\alpha} f(x)$;
(3) $\mathscr{J}^{\alpha} x^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

3. The Homotopy Analysis Method

In this section we extend the homotopy analysis method proposed by Liao [1 – 5] to differential equations with fractional derivatives.

Considering the fractional differential equation

$$\mathscr{N}_*(u(x,t)) = 0, \quad (3.1)$$

where $\mathscr{N}_*$ is a differential operator with fractional derivatives. $u(x,t)$ is an unknown function and x,t denote independent variables. For simplicity, we ignore all boundary or initial conditions. They can be treated in the same way.

Following the procedure of the HAM, we choose $u_0(x,t)$ as an initial guess of $u(x,t)$, $\mathscr{L}_*$ is a linear operator, which may contain fractional derivatives; it possesses the property $\mathscr{L}_*(0) = 0$. Thus, we can construct the zero-order deformation equation

$$\begin{aligned}&(1-q)\mathscr{L}_*(U(x,t;q) - u_0(x,t)) \\ &= qhH(x,t)\mathscr{N}_*(U(x,t;q)),\end{aligned} \quad (3.2)$$

where $q \in [0,1]$ is the embedding parameter, h and $H(x,t)$ are a nonzero auxiliary parameter and an auxiliary function, respectively. When $q = 0$ and $q = 1$, it holds

$$U(x,t;0) = u_0(x,t), \quad U(x,t;1) = u(x,t). \quad (3.3)$$

As q increases from 0 to 1, the solution $U(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Assuming that the auxiliary function $H(x,t)$ and the auxiliary parameter h are properly chosen so that $U(x,t;q)$ can be expressed by the Taylor series

$$U(x,t;q) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) q^n, \quad (3.4)$$

where

$$u_n(x,t) = \frac{1}{n!} \frac{\partial^n U(x,t,q)}{\partial q^n}\Big|_{q=0}, \; n \geq 1.$$

Besides that the above series is convergent at $q = 1$. Then, using (3.3), we have

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t). \quad (3.5)$$

For the sake of simplicity, define the vectors

$$\overrightarrow{u}_n(x,t) = \{u_0(x,t), u_1(x,t), ..., u_n(x,t)\}.$$

Differentiating the zero-order deformation (3.2) n times with respect to the embedding parameter q, then setting $q = 0$, and finally dividing by $n!$, we have the n-th-order deformation equation

$$\mathscr{L}_*[u_n(x,t) - \chi_n u_{n-1}(x,t)] = hH(x,t) R_{*_n}[\overrightarrow{u}_{n-1}(x,t)], \quad (3.6)$$

where

$$R_{*_n}[\overrightarrow{u}_{n-1}(x,t)] = \frac{1}{(n-1)!} \left\{ \frac{\partial^{n-1}}{\partial q^{n-1}} \mathscr{N}_* \left[\sum_{m=0} u_m(x,t) q^m \right] \right\} \Big|_{q=0} \quad (3.7)$$

and

$$\chi_n = \begin{cases} 0, \; n \leq 1, \\ 1, \; n > 1, \end{cases} \quad (3.8)$$

with the initial condition

$$u_n(x,t;0) = 0, \; n \geq 1. \quad (3.9)$$

Note that the higher-order deformation equation (3.6) is governed by the same linear operator $\mathscr{L}_*$. The term $R_{*_n}[\overrightarrow{u}_{n-1}(x,t)]$ can be expressed simply by $u_1(x,t), u_2(x,t), \ldots u_{n-1}(x,t)$. However, there are fractional differentiations in every higher-order deformation equation, as Caputo's differentiation is a linear operation. So we can solve the higher-order deformation equations one after the other. The N-th order approximation of $u(x,t)$ is given by

$$u(x,t) \approx u_0(x,t) + \sum_{m=1}^{N} u_m(x,t). \quad (3.10)$$

Liao [5] proved that, as long as a solution series given by the HAM converges, it must be one of the solutions. In fact, this conclusion is also satisfied for nonlinear differential equations with fractional derivatives, which can be shown as follows:

Theorem. As long as the series

$$u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)$$

is convergent, where $u_m(x,t)$ is governed by the higher-order deformation equation (3.6) under the definitions (3.7) and (3.8), it must be a solution of the original equation (3.1).

Proof: As the series $u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)$ is convergent, it holds that

$$\lim_{m\to\infty}u_m(x,t)=0.$$

Using (3.6) and (3.8), we have

$$hH(x,t)\sum_{m=1}^{+\infty}R_{*m}(\overrightarrow{u}_{m-1}(x,t)) = \sum_{m=1}^{+\infty}\mathscr{L}_*[u_m(x,t)-\chi_m u_{m-1}(x,t)].$$

Due to the linearity of the Caputo derivative, it follows

$$\sum_{m=1}^{+\infty}\mathscr{L}_*[u_m(x,t)-\chi_m u_{m-1}(x,t)] = \mathscr{L}_*[\sum_{m=1}^{+\infty}(u_m(x,t)-\chi_m u_{m-1}(x,t))] = \mathscr{L}_*[\lim_{m\to\infty}u_m(x,t)]=\mathscr{L}_*[0]=0.$$

Therefore,

$$hH(x,t)\sum_{m=1}^{+\infty}R_{*m}(\overrightarrow{u}_{m-1}(x,t))=0.$$

Since $h\neq 0$, $H(x,t)\neq 0$, we have according to the definition (3.7) that

$$\sum_{m=1}^{+\infty}R_{*m}(\overrightarrow{u}_{m-1}(x,t)) = \sum_{m=1}^{+\infty}\left[\frac{1}{(m-1)!}\left\{\frac{\partial^{m-1}}{\partial q^{m-1}}\mathscr{N}_*\left[\sum_{n=0}^{\infty}u_n(x,t)\,q^n\right]\right\}\Bigg|_{q=0}\right]=0.$$

We note, that there is no differential with respect to q in the nonlinear operator $\mathscr{N}_*$. So $\mathscr{N}_*(\sum_{n=0}^{\infty}u_n(x,t)\,q^n)$ can be looked upon as a polynomial on q. By using the binomial expansion theorem, we obtain $\mathscr{N}_*(\sum_{n=0}^{\infty}u_n(x,t))=0$, such as for $\mathscr{N}_*=u(x,t){u_x}^2$. Letting $u(x,t)=u_0(x,t)+u_1(x,t)q+u_2(x,t)q^2$, it can be easily verified that

$$\sum_{m=1}^{+\infty}\left[\frac{1}{(m-1)!}\left\{\frac{\partial^{m-1}}{\partial q^{m-1}}\mathscr{N}_*\left[\sum_{n=0}^{2}u_n(x,t)\,q^n\right]\right\}\Big|_{q=0}\right] = (u_0(x,t)+u_1(x,t)+u_2(x,t)) \cdot[(u_0(x,t)+u_1(x,t)+u_2(x,t))_x]^2 = \mathscr{N}_*(u_0(x,t)+u_1(x,t)+u_2(x,t)).$$

This ends the proof.

4. Applications of the Homotopy Analysis Method

In this section, two examples are considered to illustrate the effectiveness of the HAM for differential equations with fractional derivatives.

Example 1. Consider the nonlinear fractional Korteweg – de Vries (KdV) equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+(p+1)u^p u_x+u_{xxxx}=0,\quad t>0,\quad 0<\alpha<1, \tag{4.1}$$

with the initial condition [20]

$$u(x,0)=A[\mathrm{sech}^2(Kx)]^{1/p}, \tag{4.2}$$

where $p>0, A$ and K are constants. $\partial^\alpha/\partial t^\alpha$ is the fractional time derivative operator of order α.

This equation has a wide range of applications in plasma physics, fluid physics, capillary-gravity waves, nonlinear optics and chemical physics.

From (4.2), it is straightforward to express the solution u by a set of base functions

$$\{e_n(x)\,t^{\frac{p}{q}},\ p,q\in Z\}, \tag{4.3}$$

where $e_n(x)$ as a coefficient is a function with respect to x. This provides us with the so-called rule of solution expression.

Choosing $u_0(x,t)=u(x,0)$ and the linear operator $\mathscr{L}_*$ as

$$\mathscr{L}_*(u(x,t))=\frac{\partial^\alpha}{\partial t^\alpha},\quad 0<\alpha<1. \tag{4.4}$$

We note that there is a fractional derivative in $\mathscr{L}_*$. For simplicity, one may choose $H(x,t) = 1$ in this example. Other related formulas are the same as those given in Section 3. From (4.1) and (3.7),

$$R_{*n} = \frac{\partial^\alpha u_{n-1}}{\partial t^\alpha} + (u_{n-1})_{4x} + \frac{p+1}{(n-1)!} \cdot \left\{ \frac{\mathrm{d}^{n-1}}{\mathrm{d}q^{n-1}} \left[\left(\sum_{i=1}^{n-1} u_i q^i \right)^p \frac{\partial}{\partial x} \left(\sum_{i=1}^{n-1} u_i q^i \right) \right] \right\} \Big|_{q=0}. \quad (4.5)$$

The components of the solution series can be easily computed from (3.6) and (4.5), and we have

$$u_n = u_{n-1} - h \mathscr{J}^\alpha (R_{*n}),$$

in which $\mathscr{J}^\alpha$ is the Riemann-Liouville integral operator of order α with respect to t.

In the following, we list the first few components of the solution series for the case $p = 6$, and discuss its convergence:

$$u_1(x,t) = -\frac{t^\alpha hAK(189A^6 \sinh(Kx)\cosh(Kx) - K^3[\cosh(Kx)^4 - 200\cosh(Kx)^2 + 280])}{81\cosh(Kx)^{(13/3)}\Gamma(\alpha+1)},$$

$$\begin{aligned} u_2(x,t) = &-\frac{1}{6561}h^2 AK(455112t^{(2\alpha)}K^4\Gamma(\alpha+1)A^6\cosh(Kx)^5\sinh(Kx) \\ &-22680t^\alpha\cosh(Kx)^4\Gamma(2\alpha+1)K^3 - 5847660t^{(2\alpha)}K^4\Gamma(\alpha+1)A^6\cosh(Kx)^3\sinh(Kx) \\ &+8474760t^{(2\alpha)}K^4\Gamma(\alpha+1)A^6\cosh(Kx)\sinh(Kx) - 464373t^{(2\alpha)}K\Gamma(\alpha+1)A^{12}\cosh(Kx)^4 \\ &+16200t^\alpha\cosh(Kx)^6\Gamma(2\alpha+1)K^3 + 15309t^\alpha\cosh(Kx)^5\Gamma(2\alpha+1)A^6\sinh(Kx) \\ &-81t^\alpha\cosh(Kx)^8\Gamma(2\alpha+1)K^3 + 571536t^{(2\alpha)}K\Gamma(\alpha+1)A^{12}\cosh(Kx)^2 \\ &-24344320t^{(2\alpha)}K^7\Gamma(\alpha+1) + 33779200t^{(2\alpha)}K^7\Gamma(\alpha+1)\cosh(Kx)^2 \\ &-t^{(2\alpha)}K^7\Gamma(\alpha+1)\cosh(Kx)^8 - 11049360t^{(2\alpha)}K^7\Gamma(\alpha+1)\cosh(Kx)^4 \\ &+480400t^{(2\alpha)}K^7\Gamma(\alpha+1)\cosh(Kx)^6)\Big/(\cosh(Kx)^{(25/3)}\Gamma(2\alpha+1)\Gamma(\alpha+1)), \\ &\cdots. \end{aligned}$$

With the aid of Maple, 15 terms are used to evaluate the approximate solution $u_{\text{approx}}(x,t) = \sum_{k=0}^{15} u_k(x,t)$. We stretch the h-curve of $u''_{\text{approx}}(0,0)$ in Fig. 1, which shows that the solution series is convergent if $-0.5 \le h \le 0.2$. We take the arithmetic average value $h = -0.2$.

To verify the convergence of our approximate solution, we compare the approximate solution and the exact solution [20]

$$u_{\text{exact}} = \frac{1}{5}\,\mathrm{sech}^{1/3}\left(\frac{x}{10} - ct\right) \quad (4.6)$$

at $t = 0.1$. The graph is given in Figure 2.

Also for larger times, t, our approximate solutions are in good agreement with the exact ones, which are both shown in Figs. 3a and 3b.

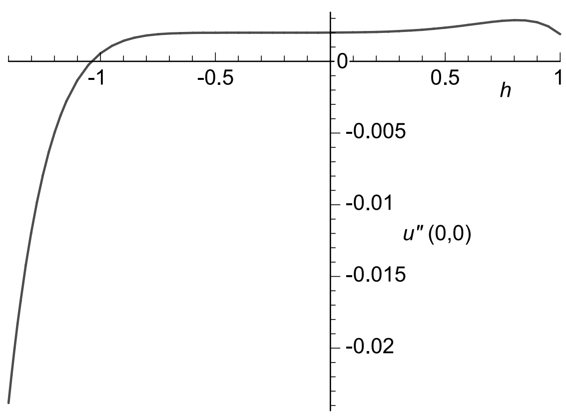


Fig. 1. The h-curve of $u''(0,0)$ at the 15th-order of approximation for fixed values $K = 0.1, A = 0.2, \alpha = 1/2$.

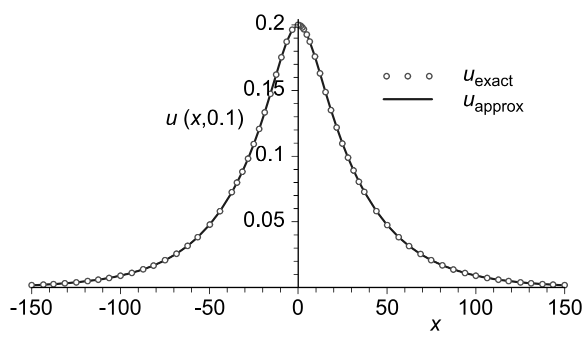
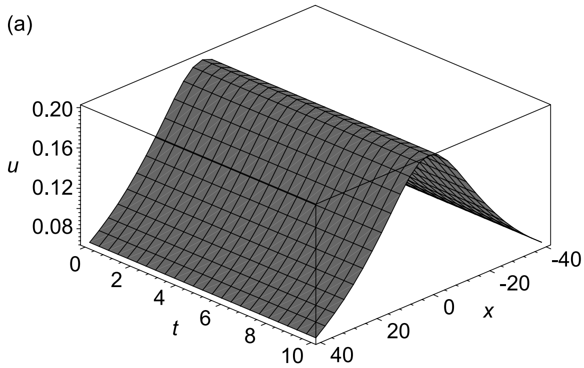


Fig. 2. Comparison of the approximate solution u_{approx} with the exact solution (4.6) at $t = 0.1$, for fixed values $K = 0.1$, $A = 0.2$, $\alpha = 1/2$.



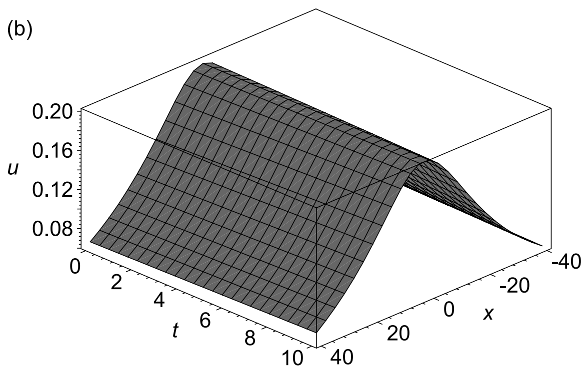


Fig. 3. (a) Approximate solution of (4.1) for fixed values $K = 0.1$, $A = 0.2$, $\alpha = 1/2$. (b) Exact solution of (4.1) for fixed values $K = 0.1$, $A = 0.2$, $\alpha = 1/2$.

Example 2. Consider the space-fractional telegraph equation [21, 22]

$$\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} - u_{2t} - u_t - u(x,t) = 0, \tag{4.7}$$

with the initial and boundary conditions

$$u(0,t) = \mathrm{e}^{-t}, \quad u_x(0,t) = \mathrm{e}^{-t}, \quad u(x,0) = \mathrm{e}^{x}. \tag{4.8}$$

From (4.8) it is straightforward to express the solution u by a set of base functions

$$\{g_n(t) x^{\frac{p}{q}},\ p,q \in Z\}, \tag{4.9}$$

where $g_n(t)$ as a coefficient is a function with respect to t. This provides us with the so-called rule of solution expression.

Choose $u_0(x,t) = \mathrm{e}^{-t}(1+x)$ and the linear operator $\mathscr{L}_*$ as

$$\mathscr{L}_*(u(x,t)) = \frac{\partial^{\lceil\alpha\rceil} u(x,t)}{\partial x^{\lceil\alpha\rceil}}, \quad 0 < \alpha \le 2. \tag{4.10}$$

In this example, there are just integer-order derivatives in $\mathscr{L}_*$. For simplicity we choose $H(x,t) = 1$. The other formulas are the same as those given in Section 3, from (3.7) and (4.7),

$$R_{*n} = \frac{\partial^{\alpha} u_{n-1}}{\partial x^{\alpha}} + (u_{n-1})_{tt} - (u_{n-1})_t - u_{n-1}. \tag{4.11}$$

The components of the solution series can be easily computed from (3.6) and (4.11), and we have

$$u_m = u_{m-1} - h \mathscr{J}^{\lceil\alpha\rceil}(u_{(m-1)_t} + u_{(m-1)_{2t}} + u_{m-1}) + h \mathscr{J}^{\lceil\alpha\rceil-\alpha}\left(u_{m-1} - \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{x^k}{k!} u_{(m-1)_{kt}}(0,t)\right),$$

where $\mathscr{J}^{\lceil\alpha\rceil}$ is the $\lceil\alpha\rceil$-fold integral with respect to x and $\mathscr{J}^{\lceil\alpha\rceil-\alpha}$ is the Riemann-Liouville integral operator of order $(\lceil\alpha\rceil - \alpha) > 0$.

For $\alpha = 2$ the first few components are:

$$u_1(x,t) = \left(\left(-\frac{1}{2}x^2 - \frac{1}{6}x^3\right) h + 1 + x\right) \mathrm{e}^{(-t)},$$

$$u_2(x,t) = \left(\left(\frac{1}{24}x^4 + \frac{1}{120}x^5 - \frac{1}{2}x^2 - \frac{1}{6}x^3\right) h^2 + \left(-x^2 - \frac{1}{3}x^3\right) h + 1 + x\right) \mathrm{e}^{(-t)},$$

$$u_3(x,t) = \left(\left(-\frac{1}{720}x^6 - \frac{1}{5040}x^7 + \frac{1}{12}x^4 + \frac{1}{60}x^5 - \frac{1}{2}x^2 - \frac{1}{6}x^3\right) h^3 + \left(\frac{1}{8}x^4 + \frac{1}{40}x^5 - \frac{3}{2}x^2 - \frac{1}{2}x^3\right) h^2 \right.$$
$$\left. + \left(-\frac{3}{2}x^2 - \frac{1}{2}x^3\right) h + 1 + x\right) \mathrm{e}^{(-t)},$$

$$u_4(x,t) = \left(\left(-\frac{1}{240}x^6 - \frac{1}{1680}x^7 + \frac{1}{8}x^4 + \frac{1}{40}x^5 + \frac{1}{40320}x^8 + \frac{1}{362880}x^9 - \frac{1}{2}x^2 - \frac{1}{6}x^3\right) h^4\right.$$

$$+\left(\frac{1}{3}x^4+\frac{1}{15}x^5-2x^2-\frac{2}{3}x^3-\frac{1}{180}x^6-\frac{1}{1260}x^7\right)h^3+\left(\frac{1}{4}x^4+\frac{1}{20}x^5-3x^2-x^3\right)h^2$$

$$+\left(-2x^2-\frac{2}{3}x^3\right)h+1+x\Bigg)\mathrm{e}^{(-t)},$$

$\cdots.$

With the help of Maple, 25 terms are used to evaluate the approximate solution $u_{\text{approx}}(x,t)=\sum_{k=0}^{25}u_k(x,t)$. The h-curve of $u''_{\text{approx}}(0,0)$ is displayed in Fig. 4, which shows that the solution series is convergent if $-1.5\le h\le -0.5$. We take the mean value $h=-1$ to get the fastest convergence rate.

To show the convergence of our solution series, the 23rd-order and the 25th-order approximate solutions are compared in Figure 5.

The 25th-order approximate solution itself is displayed in Figure 6a. To demonstrate the effectiveness of our method, we show the exact solution $u(x,t)=\mathrm{e}^{x-t}$ for $\alpha=2$ in Figure 6b. It can be seen that our approximate solution is in good agreement with the above exact solution.

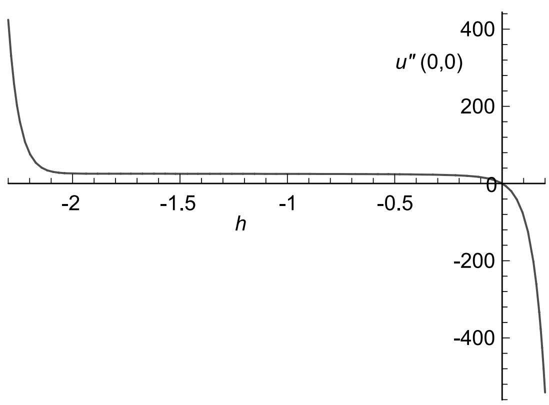


Fig. 4. The h-curve of $u''(0,0)$ at the 25th-order of approximation.

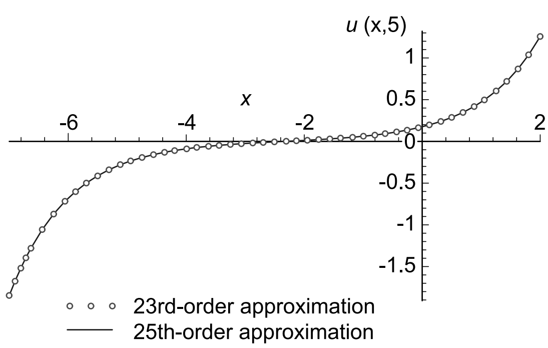


Fig. 5. Comparison of the 23rd-order with the 25th-order approximate solutions.

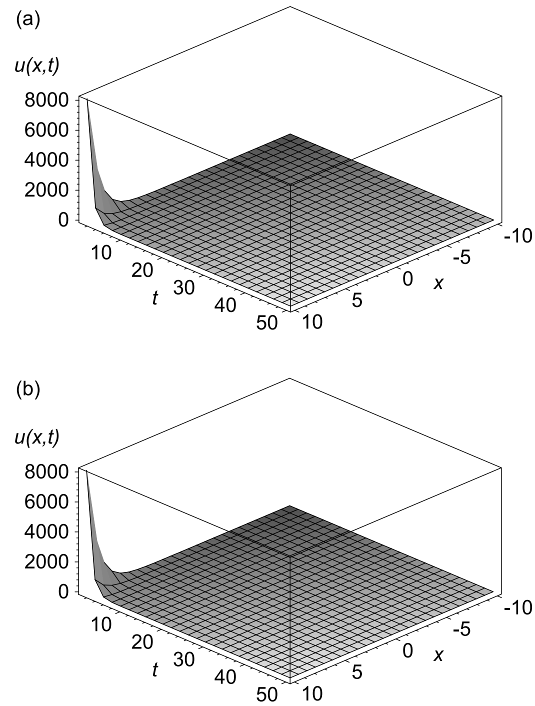


Fig. 6. (a) 25th-order approximation of (4.7). (b) Exact solution of (4.7).

5. Summary

In this paper, the HAM is employed to analytically compute approximating solutions of nonlinear differential equations with fractional derivatives. By comparing with known exact solutions, it is shown that the obtained approximate solutions have a very high accuracy. It can be seen that the HAM is effective and efficient not only for classical differential equations but also for differential equations with fractional derivatives. We help that this method can be used in a wider range.

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